- Chapter 11: Networks And Graphs
- Topic 15: Introduction to Networks and Graphs

Terminology:

- See handout for terminology: 1 – 8

Classwork Questions: Pg 201 - 204 Ex 11A Qs 3/4/6/8/9/11(evens)/14/16/17

- See handout for terminology: 9 – 13

Classwork Questions: Pg 207 - 209 Ex 11B Qs 2/4/5/7/9/10/12

- See handout for terminology: 14 – 16

Classwork Questions: Pg 211 - 212 Ex 11C Qs 1/2/4/6/7

Eulerian Cycle – uses all edges
Hamiltonian Cycle – uses all vertices

- Topic 16: Adjacency Matrices

- A matrix is a rectangular array of numbers.
- They are used in computer science to store data.
- In order for us to be able to do some calculations with networks and graphs, we can convert a graph into a matrix, using figures to represent the connections within the graph.
- An example of a matrix is shown below:

$$\begin{pmatrix} 5 & -2 \\ 3 & 6 \end{pmatrix}$$

- The dimensions of any matrix are $m \times n$ where m is the number of rows and n is the number of columns.
- The matrix shown above would be a 2x2 matrix.
- In general, an "m x n matrix" has m rows and n columns.

• Adding/Multiplying Matrices:

- When adding or multiplying matrices together, we use the following rules:

Adding 2x2 Matrices	Multiplying 2x2 Matrices
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$
Adding 3x3 Matrices	**Multiplying 3x3 Matrices**
$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} = \begin{pmatrix} a+p & b+q & c+r \\ d+s & e+t & f+u \\ g+v & h+w & i+x \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} = \begin{pmatrix} ap+bs+cv & aq+bt+cw & ar+bu+cx \\ dp+es+fv & dq+et+fw & dr+eu+fx \\ gp+hs+iv & gq+ht+iw & gr+hu+ix \end{pmatrix}$

- **Example:** If $P = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ -3 & 5 & 2 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 4 & 0 & 1 \end{pmatrix}$, find

  i) P + Q (ii) PQ (iii) RS

  Solution:

  i) $P + Q = \begin{pmatrix} 4+1 & -1+0 \\ -2+3 & 3-1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 1 & 2 \end{pmatrix}$

  ii) $PQ = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} (4)(1)+(-1)(3) & (4)(0)+(-1)(-1) \\ (-2)(1)+(3)(3) & (-2)(0)+(3)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 7 & -3 \end{pmatrix}$

  iii) $\mathrm{RS} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ -3 & 5 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 4 & 0 & 1 \end{pmatrix}$

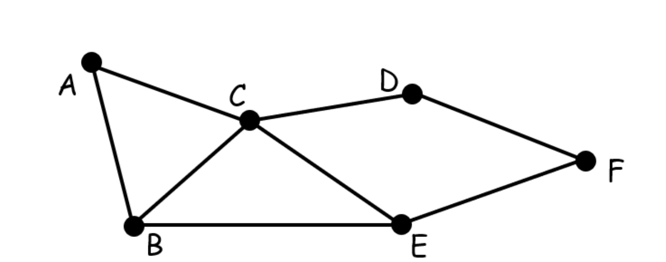
  $$= \begin{pmatrix} 0(1)+1(3)+4(-1) & 2(1)-1(3)+0(-1) & -2(1)+3(3)+1(-1) \\ 0(0)+1(2)+4(1) & 2(0)-1(2)+0(1) & -2(0)+3(2)+1(1) \\ 0(-3)+1(5)+4(2) & 2(-3)-1(5)+0(2) & -2(-3)+3(5)+1(2) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 6 & -2 & 7 \\ 13 & -11 & 23 \end{pmatrix}$$

  Note: Multiplication of matrices is not commutative, so $RS \neq SR$, for example.

Classwork Questions: Extra Sheet with Matrix Multiplication

- Adjacency Matrices:

- An adjacency matrix uses figures to represent the number of edges connecting different nodes together.
- When reading matrices, we read them left to top (Lawn Tennis). E.g. in the matrix shown below, the second row represents all the connections to node B.
- The adjacency matrix for the graph shown below is shown on the right:

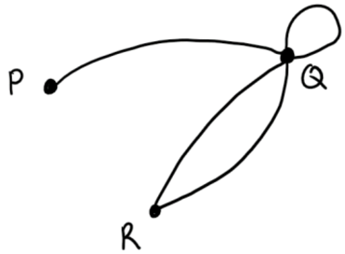


$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

- Looking at the first row, which represents the connections to A, you can see that A is connected to B and C by 1 edge each, so there are entries of '1' for those and '0' for the other nodes as A is not connected to any of those.
- You might observe also that the adjacency matrix is a square matrix 6x6 and is symmetrical about its main diagonal from top left to bottom right

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

- Example: Consider the network below. Answer the questions that follow.



Plan:
1. Start with a recap of what a “Walk” ins.
2. Show them how to write down the adjacency matrix.
3. Get them to do part (ii) themselves.
4. Get them to do parts (iii) and (iv) without using M^2 and M^3.
5. Then show how to use matrices.
6. Then let them try part (v).

(i) Write down the adjacency matrix M for this network.

(ii) Calculate M^2 and M^3.

(iii) Write down the number of walks of length 2 from P to R.

(iv) Write down the number of walks of length 3 from P to R.

(v) How many walks of length 3 from Q to P? Write down these walks.

Solution:

(i) The adjacency matrix for the graph above will be: $M = \begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \end{array}$

(ii) $$M^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{pmatrix} 1 & 1 & 2 \\ 1 & 6 & 2 \\ 2 & 2 & 4 \end{pmatrix} \end{array}$$

$$M^3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 6 & 2 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{pmatrix} 1 & 6 & 2 \\ 6 & 11 & 12 \\ 2 & 12 & 4 \end{pmatrix} \end{array}$$

Show matrix calculator

(iii) Using matrix M^2, we can see that there are 2 walks of length 2 from P to R.

(iv) Using matrix M^3, we can see that there are 2 walks of length 3 from P to R.

(v) Using matrix M^3, we can see that there are 6 walks of length 3 from Q to P.

They are: QRQP x 4, QQP, QPQP

- Powers of Adjacency Matrices:

- If A is an adjacency matrix, then the entries in A^n represent the number of walks of length n from one node to another.
- For example, for some adjacency matrix A the matrix A^3 is shown below.

$$A^3 = \begin{array}{cc} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 & 0 \end{pmatrix} \end{array}$$

2 walks of length 3 between A and B

0 walks of length 3 between D and B

5 walks of length 3 between D and F

- Key Idea:

$\boxed{A^n \text{ gives the number of walks of length n between the nodes}}$

Classwork Questions: Pg 214-216 Ex 11D Qs 1 – 3/4/6/7(i)(iii)/8

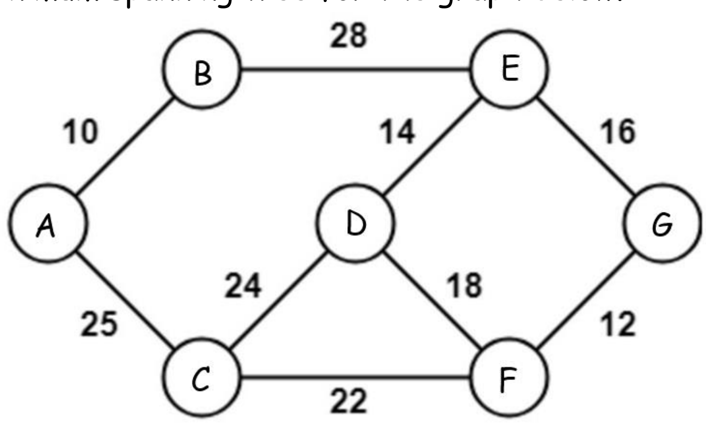
- Adjacency Matrices for Digraphs:
- Take care when dealing with digraphs (directed graphs) as they may not be symmetrical.

Classwork Questions: Pg 217/218 Ex 11E Qs 1/2/4 and 5 if needed

- Topic 17: Trees and Minimum Spanning Trees:
- See handout for terminology: 17 – 18

Classwork Questions: Pg 219/220 Ex 11F Qs 1/3/4/6/9

- Kruskal's Algorithm:
- See handout for terminology: 19
- See handout for the steps of Kruskal's Algorithm, which finds the minimum spanning tree.
- Example: Find the minimum spanning tree for the graph below.



- We begin by organising the edges in increasing order of weight:

Edge	Weight

Note: If two edges of the same weights, it doesn't matter the order we list

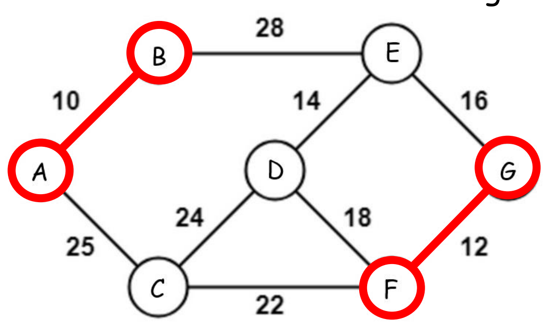
AB	10
FG	12
DE	14
EG	16
DF	18
CF	22
CD	24
AC	25
BE	28

Step 1: Pick the smallest edge.

- That'll be the first edge on our list above i.e. AB

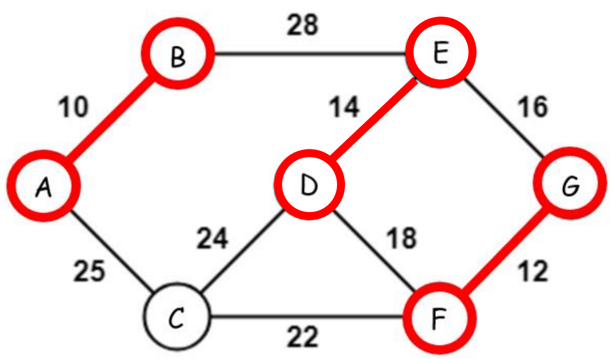
Step 2: Repeatedly look for smallest edge that does not create a cycle. If both vertices are already in the minimum spanning tree, then to not pick it – this will prevent cycles.

- The next smallest edge that we can add in without creating a cycle is FG:

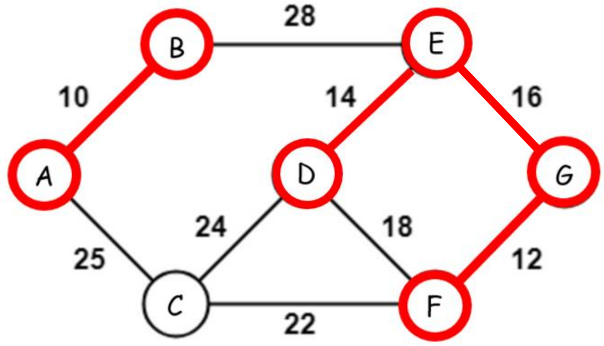


Step 3: Keep adding edges until all nodes have been selected and are in the same tree.

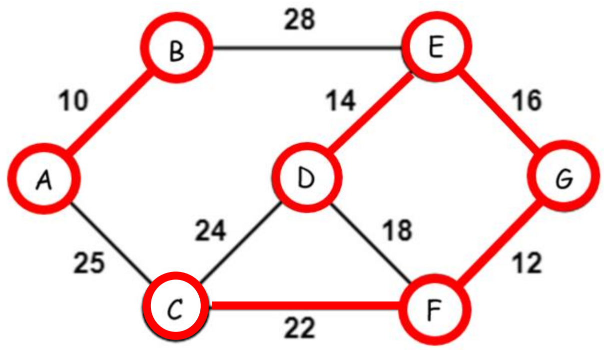
- The next smallest edge we can add on is DE:



- And then edge EG can be added:

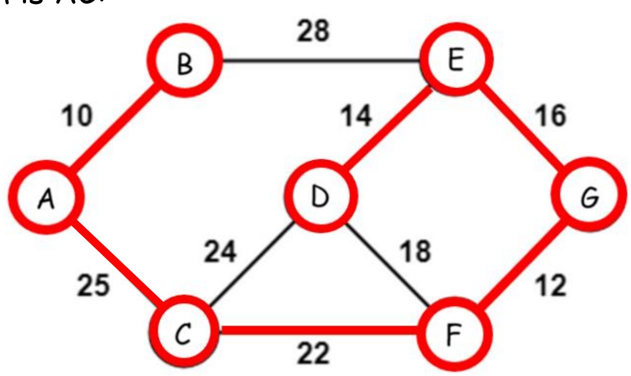


- The next smallest edge in our table is DF but if we add that to our tree, we will create a cycle, so we cross that one off and proceed to the next one, which is CF.
- As this doesn't create a cycle, we can add it:



Edge	Weight
AB	10
FG	12
DE	14
EG	16
~~DF~~	~~18~~
CF	22
CD	24
AC	25
BE	28

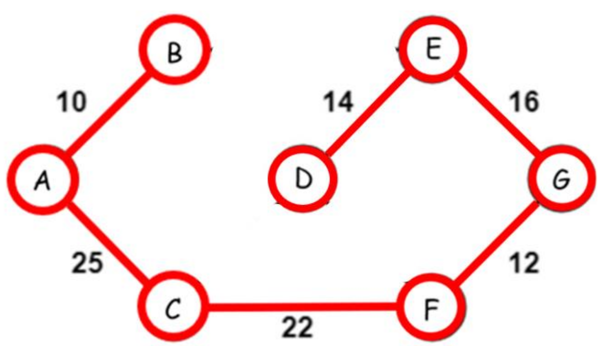
- Again, the next smallest edge CD will create a cycle so we cross that one off and add our next smallest, which is AC:



Edge	Weight
AB	10
FG	12
DE	14
EG	16
~~DF~~	~~18~~
CF	22
~~CD~~	~~24~~
AC	25
~~BE~~	~~28~~

Step 4: When all the nodes have been selected and are in the same tree, we can re-draw the nodes and the edges that we selected – they should have no cycles visible.

- Our minimum spanning tree is shown below:



Note: A useful check is that the minimum spanning tree for n nodes will have n-1 edges.

Step 5: If the edge weights are distinct the minimum spanning tree is unique. If we add up the weights of each of the edges selected, we will get the total edge weight of the minimum spanning tree.

- We can now simply add up the non-crossed-off edges from our initial table to find the weight of our minimum spanning tree:

Edge	Weight
AB	10
FG	12
DE	14
EG	16
~~DF~~	~~18~~
CF	22
~~CD~~	~~24~~
AC	25
~~BE~~	~~28~~

Note: If all edges have distinct weights i.e. they're all different, then the minimum spanning tree will be unique.

10 + 25 + 22 + 12 + 16 + 14 = 99

Note: Kruskal's algorithm is known as a greedy algorithm, as we make the decision each time to include edges, or not, regardless of what comes later.

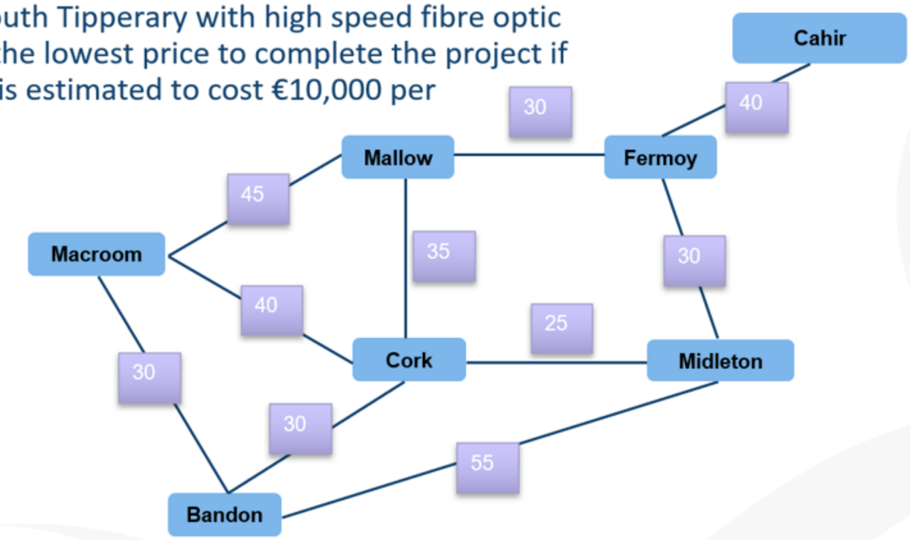
Classwork Questions: Pg 224-226 Ex 11G Qs 2/3/5/8

- Prim's Algorithm:

- Another method of finding the minimum spanning tree is to use Prim's Algorithm.
- See handout for the steps of Prim's Algorithm, which finds the minimum spanning tree.

- Example: Find the minimum spanning tree for the graph below.

A telecommunications company is contracted to connect six towns in Cork and South Tipperary with high speed fibre optic cabling. Determine the lowest price to complete the project if the fibre optic cable is estimated to cost €10,000 per Kilometre.



Step 1: Create an empty list to keep track of the nodes we have touched.

Nodes Visited	Cumulated Weight

Step 2: Pick an arbitrary node, say Mallow.

Step 3: Add Mallow to the visited list. Now look at all Nodes reachable from Mallow and select the smallest edge from Mallow that connects to an unvisited node

- The smallest weight is 30, which connects in Fermoy.

Step 4: Add Fermoy to the visited list.

Nodes Visited	Cumulated Weight
Mallow	.-
Fermoy	30

Step 5: Now look at all the nodes reachable from Mallow and Fermoy, select the smallest edge that connects to an unvisited node.

- The smallest weight connecting to either Mallow or Fermoy is 30, which brings in Midleton.

Nodes Visited	Cumulated Weight
Mallow	.-
Fermoy	30
Midleton	30 + 30 = 60

Step 6: Continue in this manner each time selecting the smallest edge connecting to an unvisited node. (Note: Make sure you don't create any cycles when adding edges)

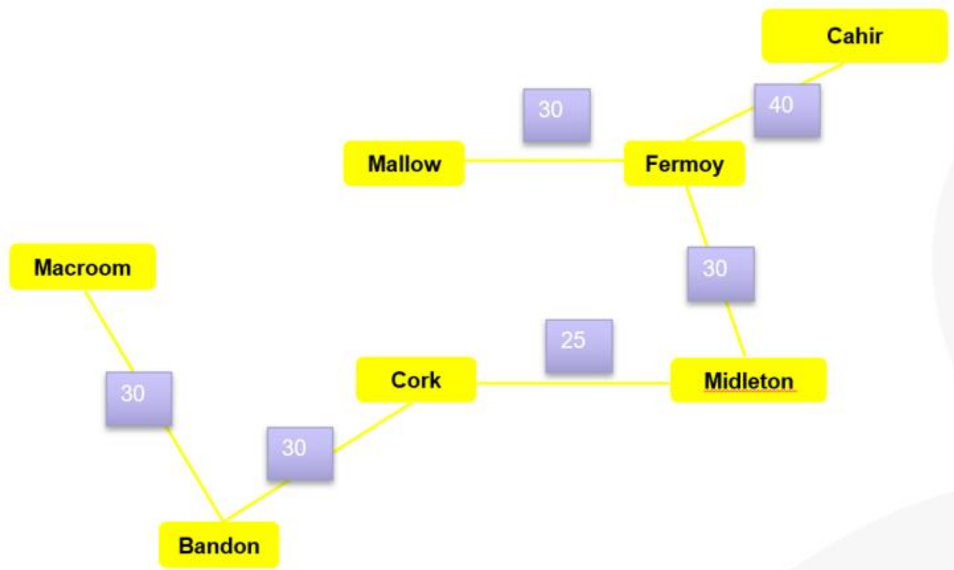
Nodes Visited	Cumulated Weight
Mallow	.-
Fermoy	30

Midleton	30 + 30 = 60
Cork	60 + 25 = 85
Bandon	85 + 30 = 115
Macroom	115 + 30 = 145
Cahir	145 + 40 = 185

Step 7: If both vertices are already in the minimum spanning tree, then do not pick it – this will prevent cycles.

- As long as we take care to not include any cycles as we're going along, we can ignore step 7.

Step 8: When all the nodes have been visited, we can re-draw the nodes and the edges that we selected – they should have no cycles visible.



Step 9: If the edge weights are distinct the minimum spanning tree is unique. If we add up the weights of each of the edges selected, we will get the total edge weight of the minimum spanning tree.

- There are multiple edges with the same weights so this minimum spanning tree we found isn't unique.
- The combined weight of this one from our table above is 185 km.
- Referring back to the original question, we were told the cost was €10,000 per kilometre, so the final cost would be 185 x 10,000 = €1,850,000

Classwork Questions : Pg 228/229 Ex 11H Qs 1/2/4

- Prim's Algorithm using a Distance Matrix:

- It's also possible to use Prim's Algorithm if a network was given as a distance matrix rather than a diagram.

- Example: Use Prim's Algorithm to find the MST, starting at C for the following distance matrix.

	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
C	5	-	-	1	4
D	-	3	1	-	2
E	4	5	4	2	-

<u>Solution:</u>

Step 1A: Again, we choose an arbitrary node to start with, say C.

Step 1B: Delete the row of the letter chosen and write a 1 over that column.

			1		
	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
D	-	3	1	-	2
E	4	5	4	2	-

Step 1C: Look down the column C and choose the least weight i.e. the 1 that connects to D, so our first edge is CD.

			1		
	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
D	-	3	1	-	2
E	4	5	4	2	-

Step 2A: As we chose D, we now delete row D and write a 2 over column D.

			1	2	
	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
E	4	5	4	2	-

Step 2B: Look down both numbered columns C and D, and choose the least weight, which is not crossed out i.e. the 2 that connects D to E, so DE is our next edge

			1	2	
	A	B	C	D	E
A	-	3	5	-	4

B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
E	4	5	4	2	-

Step 3A: As we chose E, we now delete row E and write a 3 over column E.

			1	2	3
	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
~~E~~	~~4~~	~~5~~	~~4~~	2	~~-~~

Step 3B: Look down all numbered columns, and choose the least weight, which is not crossed out i.e. the 3 that connects D to B, so DB is our next edge

			1	2	3
	A	B	C	D	E
A	-	3	5	-	4
B	3	-	-	3	5
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
~~E~~	~~4~~	~~5~~	~~4~~	2	~~-~~

Step 4A: As we chose B, we now delete row B and write a 4 over column B.

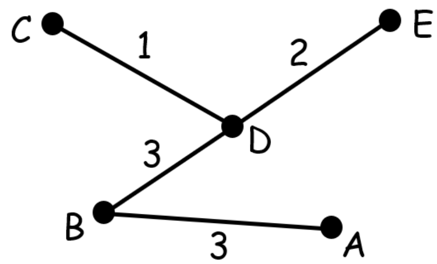
		4	1	2	3
	A	B	C	D	E
A	-	3	5	-	4
~~B~~	~~3~~	~~-~~	~~-~~	3	~~5~~
~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
~~E~~	~~4~~	~~5~~	~~4~~	2	~~-~~

Step 4B: Look down all numbered columns, and choose the least weight, which is not crossed out i.e. the 3 that connects B to A, so BA is our final edge

		4	1	2	3
	A	B	C	D	E
A	-	3	5	-	4
~~B~~	~~3~~	~~-~~	~~-~~	3	~~5~~

~~C~~	~~5~~	~~-~~	~~-~~	~~1~~	~~4~~
~~D~~	~~-~~	~~3~~	1	~~-~~	~~2~~
~~E~~	~~4~~	~~5~~	~~4~~	2	~~-~~

- We now have the 4 edges in our minimum spanning tree, with a combined weight of 3 + 3 + 1 + 2 = 9.



Classwork Questions: Pg 229/230 Ex 11H Qs 6/7/8
Revision Questions and Test