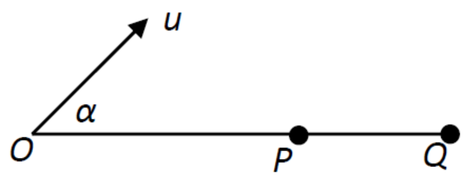
Sample Paper 6 Marking Scheme

Question 1

(a)

A particle is projected from a point *O* with speed *u* m s^{-1} at an angle α to the horizontal.

(i) Show that the range of the particle is $\frac{u^2 \sin 2\alpha}{g}$, and that the maximum range $|OQ|$ is $\frac{u^2}{g}$.



If the angle of projection is increased to 60° the particle strikes the horizontal plane at *P*.

(ii) Find the distance $|PQ|$ in terms of *u*.

(i)

$$r_j = 0 \qquad (5)$$

$$u \sin\alpha \times t - \frac{1}{2}gt^2$$

$$t = \frac{2u\sin\alpha}{g} \qquad (5)$$

$$\text{Range} = u\cos\alpha \times t$$

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g}$$

$$= \frac{u^2\sin 2\alpha}{g} \qquad (5)$$

$$|OQ| = \frac{u^2 \times 1}{g} = \frac{u^2}{g} \qquad (5)$$

(ii)

$$|OP| = \frac{u^2 \times \sin 120}{g} = \frac{u^2\sqrt{3}}{2g}$$

$$|PQ| = \frac{u^2}{g} - \frac{u^2\sqrt{3}}{2g}$$

$$= \frac{0.134u^2}{g} \text{ or } 0.014u^2 \text{ or } \frac{(2-\sqrt{3})u^2}{2g} \qquad (5) \qquad (25)$$

(b)

If there were no emigration, the population x of a certain county would increase at a constant rate of 2·5% per annum. By emigration the county loses population at a constant rate of n people per annum.
When the time is measured in years then $\frac{dx}{dt} = \frac{x}{40} - n$.

(i) If initially the population is P people, find in terms of n, P and t, the population after t years.

(ii) Given that $n = 800$ and $P = 30000$, find the value of t when the population is 29734.

(i) $$\frac{dx}{dt} = \frac{x}{40} - n$$

$$40 \times \int \frac{dx}{x-40n} = \int dt$$ (5)

$$40 \times \ [\ln(x - 40n)]_P^x = [t]_0^t$$ (5), (5)

$$40ln(x - 40n) - 40ln(P - 40n) = t$$

$$\ln\left(\frac{x-40n}{P-40n}\right) = \frac{t}{40}$$

$$x = (P - 40n)e^{t/40} + 40n$$ (5)

(ii) $$x = (P - 40n)e^{t/40} + 40n$$

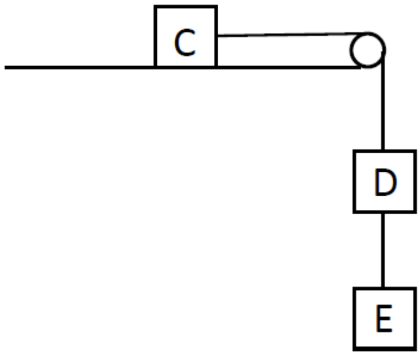
$$29734 = (30000 - 32000)e^{t/40} + 32000$$

$$2000 \times e^{t/40} = 2266$$

$$t = 40\ln(1.133)$$

$$t = 4.99 \ \text{years}$$ (5) (25)

Question 2
(a)



A block C of mass 6*m* rests on a rough horizontal table.

It is connected by a light inextensible string which passes over a smooth fixed pulley at the edge of the table to a block D of mass 3*m*. D is connected by another light inextensible string to a block E of mass 2*m*, as shown in the diagram.

The coefficient of friction between C and the table is $\frac{1}{3}$.

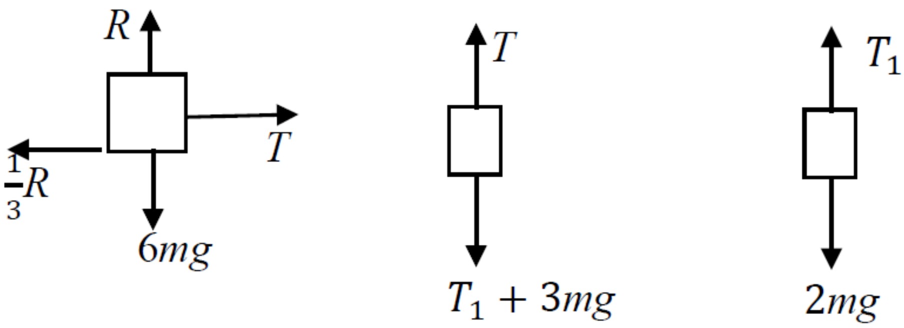
The system is released from rest.

(i) Show on separate diagrams the forces acting on each block.

(ii) Find the acceleration of C.

(iii) Find the tension in each string.

(i)



(5)

(ii)

$$T - 2mg = 6ma \qquad (5)$$

$$T_1 + 3mg - T = 3ma \qquad (5)$$

$$2mg - T_1 = 2ma \qquad (5)$$

$$a = \frac{3g}{11} \qquad (5)$$

(iii)

$$T = 2mg + 6m \times \frac{3g}{11} \quad \Rightarrow \quad T = \frac{40}{11}mg$$

$$T_1 = 2mg - 2m \times \frac{3g}{11} \quad \Rightarrow \quad T_1 = \frac{16}{11}mg \qquad (5) \qquad (30)$$

(b)

(i) $W = \int_0^h \frac{mgR^2}{(R+x)^2} dx = mgR^2 \left(-\frac{1}{R+x}\right)\Big|_0^h = \frac{mgh}{1+\frac{h}{R}}$ Q.E.D. (5) + (5)

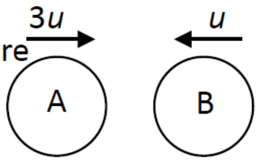
(ii) If *h* is small compared to $R, \frac{h}{R} \approx 0$ (5)

$W = \frac{mgh}{1+0} = mgh$ Q.E.D. (5)

Question 3

(a)

A smooth sphere A of mass $2m$, moving with speed $3u$ on a smooth horizontal table collides directly with a smooth sphere B of mass m, moving in the opposite direction with speed u.



The coefficient of restitution between A and B is e.

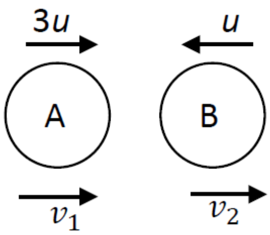
Find, in terms of u and e,

(i) the speed of each sphere after the collision

(ii) the magnitude of the impulse imparted to B due to the collision.

The loss of the kinetic energy due to the collision is $kmu^2(1 - e^2)$.

(iii) Find the value of k.



(i) PCM $\quad 2m(3u) + m(-u) = 2mv_1 + mv_2$ (5)

NEL $\quad v_1 - v_2 = -e\big(3u - (-u)\big)$ (5)

$$2v_1 + v_2 = 5u$$

$$v_1 - v_2 = -4eu$$

$$v_1 = \frac{u(5-4e)}{3} \qquad v_2 = \frac{u(5+8e)}{3}$$ (5), (5)

(ii)
$$I = \left| m\frac{u(5+8e)}{3} - m(-u) \right|$$

$$= \frac{8mu}{3}(1 + e)$$ (5)

(iii)
$$\mathrm{KE_B} = \tfrac{1}{2}(2m)(3u)^2 + \tfrac{1}{2}(m)(-u)^2 = \tfrac{19}{2}mu^2$$

$$\mathrm{KE_A} = \tfrac{1}{2}(2m)(v_1)^2 + \tfrac{1}{2}(m)(v_2)^2$$

$$= \tfrac{1}{9}mu^2\left\{(25 - 40e + 16e^2) + \tfrac{1}{2}(25 + 80e + 64e^2)\right\}$$

$$= \tfrac{1}{9}mu^2\{37.5 + 48e^2\}$$

$$\mathrm{KE_L} = \tfrac{19}{2}mu^2 - \tfrac{1}{9}mu^2\{37.5 + 48e^2\}$$

$$= \tfrac{16}{3}mu^2(1 - e^2)$$

$$\Rightarrow \quad k = \frac{16}{3}$$ (5) (30)

(b)

(i) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (5)

(ii) $\begin{pmatrix} 9 & 12 & 2 \\ 12 & 4 & 5 \\ 2 & 5 & 0 \end{pmatrix}$

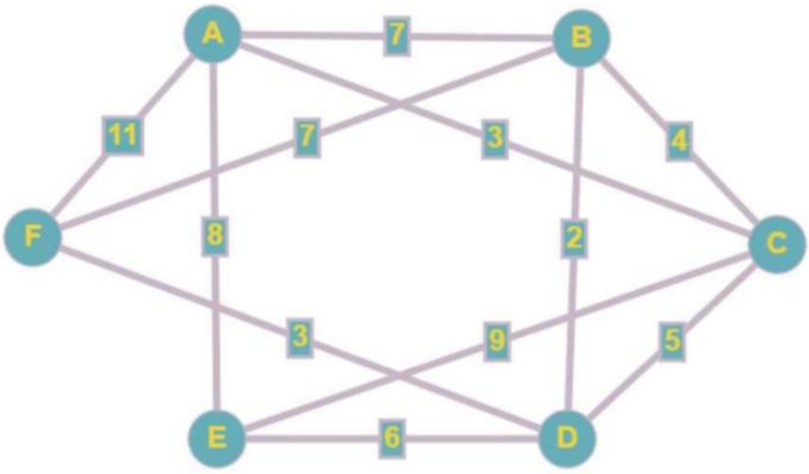
(5) for M^2 correct → $\begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

(5) for M^3 correct

(iii) 12 (5)

Question 4
(a)

(i)

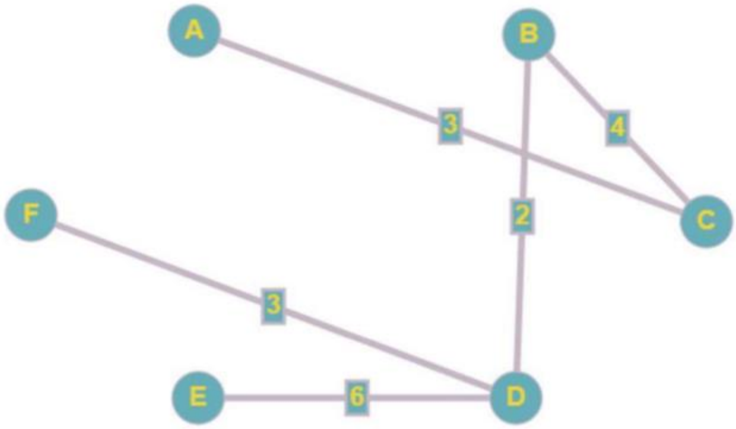


(5)

(ii) BD, AC, DF, BC, DE (5) + (5)

Length $= 18$ km

(iii)



(5)

(iv) (B, D), (D, F), (B, C), (A, C), (D, E) (5)

(b)

A particle is projected with speed $\sqrt{\frac{9gh}{2}}$ from a point *P* on the top of a cliff of height h. It strikes the ground a horizontal distance $3h$ from *P*.

(i) Find the two possible angles of projection.

(ii) For each angle of projection find, in terms of h, the time it takes the particle to reach the ground.

(i)

$$r_{\vec{i}} = 3h \qquad 5$$

$$u\cos\alpha \times t = 3h \qquad 5$$

$$t = \frac{3h}{u\cos\alpha}$$

$$r_{\vec{j}} = -h$$

$$u\sin\alpha \times t - \tfrac{1}{2}gt^2 = -h$$

$$u\sin\alpha\left(\frac{3h}{u\cos\alpha}\right) - \tfrac{1}{2}g\left(\frac{3h}{u\cos\alpha}\right)^2 = -h \qquad 5$$

$$\tan^2\alpha - 3\tan\alpha = 0$$

$$\tan\alpha = 0 \quad \Rightarrow \alpha = 0°$$

$$\tan\alpha = 3 \quad \Rightarrow \alpha = 71.6°$$

(ii)

$$t_0 = 0 \quad \text{or} \quad t_0 = \frac{3h}{u\cos\alpha} \qquad 5$$

$$= \frac{3h}{\sqrt{\frac{9gh}{2}}\cos 0} = \sqrt{\frac{2h}{g}}$$

$$t_3 = 0 \quad \text{or} \quad t_3 = \frac{3h}{u\cos\alpha}$$

$$= \frac{3h}{\sqrt{\frac{9gh}{2}}\cos 71.6} = \sqrt{\frac{20h}{g}} \qquad 5$$

25

Question 5

(a)

A ball is thrown vertically downwards from the top of a building of height *h* m. The ball passes the top half of the building in 1.2 s and takes a further 0.8 s to reach the bottom of the building.

Find

(i) the value of *h*

(ii) the speed of the ball at the bottom of the building.

(i)

$$s = ut + \frac{1}{2}at^2$$
$$\frac{1}{2}h = 1.2u + \frac{1}{2}g \times 1.44$$
$$h = 2.4u + 1.44g \qquad (5)$$

$$s = ut + \frac{1}{2}at^2$$
$$h = 2u + \frac{1}{2}g \times 4$$
$$h = 2u + 2g \qquad (5)$$

$$5h = 12u + 70.56$$
$$6h = 12u + 117.6$$

$$h = 47.04 \qquad (5)$$

(ii)

$$h = 2u + 2g$$
$$47.04 = 2u + 19.6$$
$$u = 13.72 \qquad (5)$$

$$v = u + at$$
$$v = 13.72 + 9.8 \times 2$$

$$v = 33.32 \text{ m s}^{-1} \qquad (5) \qquad (25)$$

(b)

(i) Assume: $P_n = an + b$

$20(an+b) - 28(a(n-1)+b) + 9(a(n-2)+b) = 20(40n+500)$ (5)

$a = 800,\ b = 2000$

Particular solution: $P_n = 800n + 2000$ (5)

Total solution: $P_n = l(0.9)^n + m(0.5)^n + 800n + 2000$ (5)

$3200 = l(0.9)^0 + m(0.5)^0 + 800(0) + 2000$

$2690 = l(0.9)^1 + m(0.5)^1 + 800(1) + 2000$

$l = -1775,\ m = 2975$

$P_n = 2975(0.5)^n - 1775(0.9)^n + 800n + 2000$ (5)

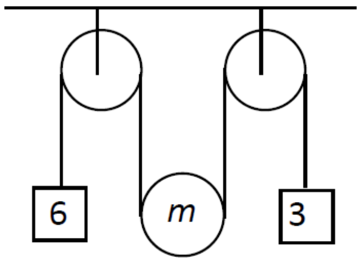
(ii) $P_{10} = 2975(0.5)^{10} - 1775(0.9)^{10} + 800(10) + 2000 \approx 9384$ (5)

Question 6

(a)

A moveable pulley of mass *m* is suspended on a light inextensible string between two fixed pulleys as shown in the diagram. Masses of 6 kg and 3 kg are attached to the ends of the string.

The system is released from rest.

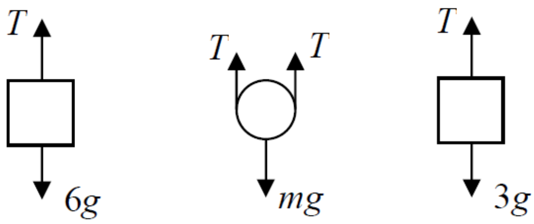


(i) Show, on separate diagrams, the forces acting on the moveable pulley **and** on each of the masses.

(ii) Find in terms of *m* the tension in the string.

(iii) For what value of *m* will the acceleration of the moveable pulley be zero?

(i)



(5)

(ii) $T - 6g = 6a$ (5)

$T - 3g = 3b$ (5)

$mg - 2T = m \times \frac{1}{2}(a + b)$

$mg - 2T = m \times \frac{1}{2}\left(\frac{T}{6} - g + \frac{T}{3} - g\right)$

$4mg - 8T = m \times (T - 4g)$

$T = \frac{8mg}{8+m}$ (5)

(iii) $\frac{1}{2}(a + b) = 0$

$mg - 2T = 0$

$mg = 2T = \frac{16mg}{8+m}$

$m = 8$ (5) (25)

(b)

A car C moves with uniform acceleration a from rest to a maximum speed u.
It then travels at uniform speed u.
Just as car C starts, it is overtaken by a car D moving in the same direction with constant speed $\frac{3u}{4}$.
Car C catches up with car D when car C has travelled a distance d.

(i) Show that, at the instant car C catches up with car D, car C has been travelling with speed u for a time $\frac{4d}{3u} - \frac{u}{a}$.

(ii) Find d in terms of u and a.

(i) D

$$s = ut + \frac{1}{2}at^2$$

$$d = \frac{3}{4}ut + 0 \qquad (5)$$

$$t = \frac{4d}{3u}$$

C

$$v = u + at$$

$$u = 0 + at_1 \qquad (5)$$

$$t_1 = \frac{u}{a}$$

$$t_2 = t - t_1 = \frac{4d}{3u} - \frac{u}{a} \qquad (5)$$

(ii) C

$$d = 0 + \frac{1}{2}at_1{}^2 + ut_2 + 0$$

$$d = \frac{1}{2}a\left(\frac{u}{a}\right)^2 + u\left(\frac{4d}{3u} - \frac{u}{a}\right) \qquad (5)$$

$$d = \frac{u^2}{2\text{a}} + \frac{4d}{3} - \frac{u^2}{\text{a}}$$

$$d = \frac{3u^2}{2\text{a}} \qquad (5) \qquad (25)$$

Question 7

(a)

A particle moves in a horizontal line such that its speed v at time t is given by the differential equation

$$\frac{dv}{dt} = 5 - 8e^{-t}.$$

(i) Given that $v = 2$ when $t = 0$, find an expression for v in terms of t.

(ii) Find the minimum value of v.

(iii) Find the distance travelled by the particle before it attains its minimum speed.

(i) $\frac{dv}{dt} = 5 - 8e^{-t}$

$\int dv = \int (5 - 8e^{-t})dt$ (5)

$[v]_2^v = [5t + 8e^{-t}]_0^t$ (5)

$v - 2 = (5t + 8e^{-t}) - 8$

$v = 5t + 8e^{-t} - 6$ (5)

(ii) $\frac{dv}{dt} = 0$

$5 - 8e^{-t} = 0$

$t = \ln\frac{8}{5} = 0.47$

$v_{\text{min}} = 5 \times 0.47 + 5 - 6 = 1.35$ (5)

(iii) $\frac{ds}{dt} = 5t + 8e^{-t} - 6$

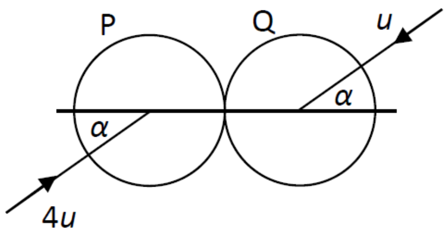
$[s]_0^s = \left[\frac{5}{2}t^2 - 8e^{-t} - 6t\right]_0^{0.47}$

$s = \left(\frac{5}{2}(0.47)^2 - 8e^{-0.47} - 6(0.47)\right) - (-8)$

$s = 0.73$ (5) (25)

(b)

A small smooth sphere P, of mass $2m$, moving with speed $4u$, collides obliquely with an equal smooth sphere Q, of mass $3m$, moving with speed u.
Before the collision the spheres are moving in opposite directions, each making an angle α to the line of centres, as shown in the diagram.
The coefficient of restitution between the spheres is $\frac{1}{5}$.



(i) Find, in terms of u and α, the speed of each sphere after the collision.

After the collision the speed of P is twice the speed of Q.

(ii) Find the value of α.

P	2m	$4u\cos\alpha\ \vec{i} + 4u\sin\alpha\ \vec{j}$	$v_1\ \vec{i} + 4u\sin\alpha\ \vec{j}$
Q	3m	$-u\cos\alpha\ \vec{i} - u\sin\alpha\ \vec{j}$	$v_2\ \vec{i} - u\sin\alpha\ \vec{j}$

(i) PCM $\quad 2m(4u\cos\alpha) + 3m(-u\cos\alpha) = 2mv_1 + 3mv_2$ (5)

NEL $\quad v_1 - v_2 = -\frac{1}{5}(4u\cos\alpha + u\cos\alpha)$ (5)

$$2v_1 + 3v_2 = 5u\cos\alpha$$

$$v_1 - v_2 = -u\cos\alpha$$

$v_1 = \frac{2}{5}u\cos\alpha \qquad v_2 = \frac{7}{5}u\cos\alpha$ (5)

Speed of P $= \sqrt{\left(\frac{2}{5}u\cos\alpha\right)^2 + (4u\sin\alpha)^2}$

Speed of Q $= \sqrt{\left(\frac{7}{5}u\cos\alpha\right)^2 + (-u\sin\alpha)^2}$ (5)

(ii) $\frac{4}{25}u^2\cos^2\alpha + 16u^2\sin^2\alpha = 4\left\{\frac{49}{25}u^2\cos^2\alpha + u^2\sin^2\alpha\right\}$

$$\tan^2\alpha = \frac{16}{25}$$

$\alpha = 38.66°$ (5) (25)

Question 8

(a)

A car passes four collinear markers *A*, *B*, *C*, and *D* while moving in a straight line with uniform acceleration. The car takes *t* seconds to travel from *A* to *B*, *t* seconds to travel from *B* to *C* and *t* seconds to travel from *C* to *D*.

If $|AB| + |CD| = k|BC|$, find the value of *k*.

$$|AB| = ut + \frac{1}{2}at^2$$ 5

$$|AC| = 2ut + \frac{1}{2}a(2t)^2 = 2ut + 2at^2$$

$$|AD| = 3ut + \frac{1}{2}a(3t)^2 = 3ut + \frac{9}{2}at^2$$

$$|BC| = ut + \frac{3}{2}at^2$$ 5

$$|CD| = ut + \frac{5}{2}at^2$$ 5

$$|AB| + |CD| = 2ut + 3at^2$$ 5

$$= 2\left\{ut + \frac{3}{2}at^2\right\}$$

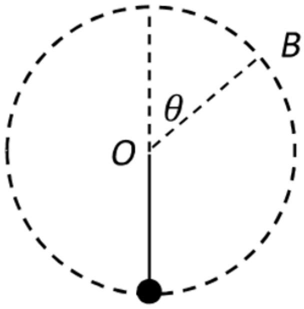
$$= 2 \times |BC|$$

$$k = 2$$ 5

25

(b)

A particle P is attached to one end of a light inextensible string of length d.
The other end of the string is attached to a fixed point O.
The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed $\sqrt{3gd}$.
The particle moves in a vertical circle.
The string becomes slack when P is at the point B.
OB makes an angle θ with the upward vertical.



(i) Show that $\cos\theta = \frac{1}{3}$.

(ii) In terms of d, find the greatest height of P above B in the subsequent motion.

(i)

$$T + mg\cos\theta = \frac{mv^2}{d} \qquad (5)$$

$$\frac{1}{2}m(3gd) = \frac{1}{2}mv^2 + mg(d + d\cos\theta) \qquad (5)$$

$$mg(1 - 2\cos\theta) = \frac{mv^2}{d}$$

$$T + mg\cos\theta = \frac{mv^2}{d} = mg(1 - 2\cos\theta)$$

$$T + 3mg\cos\theta = mg$$

$$T = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{3} \qquad (5)$$

(ii)

$$T + mg\cos\theta = \frac{mv^2}{d}$$

$$0 + mg \times \frac{1}{3} = \frac{mv^2}{d}$$

$$v^2 = \frac{1}{3}dg \qquad (5)$$

$$v^2 = u^2 + 2as$$

$$0 = (v\sin\theta)^2 - 2gs$$

$$0 = \frac{1}{3}dg \times \frac{8}{9} - 2gs$$

$$s = \frac{4d}{27} \qquad (5) \qquad (25)$$